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# Large-$N$ expansion, conformal field theory and renormalization-group flows in three dimensions

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Abstract: I study a class of interacting conformal field theories and conformal windows in three dimensions, formulated using the Parisi large-$N$ approach and a modified dimensional-regularization technique. Bosons are associated with composite operators and their propagators are dynamically generated by fermion bubbles. Renormalization-group flows between pairs of interacting fixed points satisfy a set of non-perturbative $g \leftrightarrow 1/g$ dualities. There is an exact relation between the beta function and the anomalous dimension of the composite boson. Non-abelian gauge fields have a non-renormalized and quantized gauge coupling, although no Chern-Simons term is present. A problem of the naive dimensional-regularization technique for these theories is uncovered and removed with a non-local, evanescent, non-renormalized kinetic term. The models are expected to be a fruitful arena for the study of odd-dimensional conformal field theory.

Keywords: Renormalization Regularization and Renormalons, 1/N Expansion.

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Unexpected remnants of the renormalization algorithm in quantum field theory are the Adler-Bell-Jackiw anomalies [1, 2], finite amplitudes arising from the quantum violation of classical conservation laws. Anomalies fall into two main classes: axial and trace. The axial anomalies obey all-order properties, such as the Adler-Bardeen theorem [3] , and give important information about the low-energy physics, by means of the 't Hooft anomaly-matching conditions [4]. The trace anomaly is related to the beta function [5], by a formula $\Theta = \beta^a \mathcal{O}^a$, $\mathcal{O}^a$ being composite operators. Certain trace anomalies in external fields can be computed exactly in the IR limit of (supersymmetric) UV-free theories [6, 7], where an exact beta function can also be derived [8]. They reveal the intrinsic irreversibility of the renormalization-group flow, its relation to the invariant area of the graph of the beta function between the fixed points [9] and the essential difference between marginal and relevant deformations [10].

Most of these powerful results apply only to even dimensions. Trace anomalies in external gravitational and flavour fields do not exist in odd dimensions. Nevertheless, an odd-dimensional formula for the irreversibility of the RG flow can in principle be written [11], because the relation $\Theta = \beta^a \mathcal{O}^a$ is completely general and so is the notion of invariant area of the graph of the beta function. It would be desirable to dispose of a web of non-trivial conformal field theories, conformal windows and RG flows in three dimensions to investigate these and related issues more closely. The purpose of this letter is to construct a large class of such theories and flows, and address the search for appropriate odd-dimensional generalizations of the properties mentioned above.

The beta functions of the most general power-counting renormalizable three-dimensional theory with a Chern-Simons vector field have been studied by Avdeev et al. in ref. [12]. Three-dimensional quantum field theory is relevant for its possible applications in the domain of condensed-matter physics. However, the Chern-Simons models are parity-violating and this somewhat limits the range of their applicability. The three-dimensional $\phi^6$ theory is known to have a non-trivial fixed point in the large-$N$ expansion and a conformal window interpolating between the free limit and this point [13]. Nevertheless, a large class of parity-preserving conformal windows is not known at present and will be constructed here.

The Chern-Simons coupling $g_{\rm cs}$ is not renormalized [14, 15, 16]. The simplest argument to prove this fact proceeds as follows. Let us denote by $\beta_{\rm cs}$ the beta function of $g_{\rm cs}$. The results of refs. [5], relating the trace anomaly to the beta functions, imply that, in our case, $\Theta$ should contain a term proportional to the Chern-Simons form, multiplied by $\beta_{\rm cs}$. However, $\Theta$ is gauge-invariant, while the Chern-Simons form is not. For this reason $\beta_{\rm cs}$ has to be identically zero. This kind of argument, essentially based on the properties of the trace anomaly, will be applied several times in this paper.

It was shown in ref. [12] that the Chern-Simons coupling can be used to split the zeros of the beta function and generate a variety of non-trivial conformal windows.



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For example, the beta function of a $\bar{\varphi}\varphi\bar{\psi}\psi$-coupling with constant $\eta$ typically reads

$$\beta_\eta = a(\eta + bg_{\rm cs}^2)(\eta^2 - g_{\rm cs}^4)$$

to the lowest order, $a$ and $b$ being some factors, possibily depending on the gauge group and the representation. The coupling $g_{\rm cs}$ is quantized in the non-abelian case, $g_{\rm cs}^2 = 1/N$, and is arbitrarily small in the large-$N$ limit. Therefore, the existence of interacting fixed points at $\eta = \pm g_{\rm cs}^2$ and $\eta = -bg_{\rm cs}^2$ is proved, in this limit, directly from perturbation theory. This construction is a three-dimensional analogue of the existence proof of a conformal window in QCD. There we have, to two loops,

$$\beta_{\rm QCD} = \beta_1\alpha^2 + \beta_2\alpha^3 + \mathcal{O}(\alpha^4)\,, \qquad \beta_1 = -\frac{1}{6\pi}(11N_c - 2N_f)\,, \qquad \beta_2 = \frac{25N_c^2}{(4\pi)^2}\,,$$

where $\beta_2$ is written for $\beta_1 \ll N_c$ and $N_c$ large. The role of $g_{\rm cs}$ is here played by $\beta_1/\beta_2 \ll 1/N_c$. We see that all these constructions involve a large-$N$ limit of some sort. Our models will not be an exception in this respect.

The successful removal of divergences in quantum field theory is not restricted to the power-counting renormalizable theories. Non-renormalizable models in less than four dimensions were quantized long ago by Parisi, using a large-$N$ expansion [17]. The four-fermion model has been studied in detail [18, 19], and the technique has been applied to other cases, such as the $S_{N-1}$ non-linear $\sigma$-model [20] and the $CP^{N-1}$ model [21]. A challenging, open problem in quantum field theory is to classify the set of power-counting non-renormalizable theories that can be constructed in a perturbative sense, i.e. the appropriate generalization of the power-counting criterion [22].

For the purposes of this paper, the Parisi large-$N$ expansion is a powerful tool to construct non-trivial conformal field theories and conformal windows in three dimensions. The known four-fermion models are relevant perturbations of a certain subclass of these fixed points. Our models are power-counting non-renormalizable, because although they do not contain dimensionful parameters, certain bosonic fields do not have a propagator at the classical level. Such fields are associated with composite operators and can be scalars, but also abelian and non-abelian gauge vectors. The propagators are dynamically generated by fermion loops and the large-$N$ expansion is crucial to justify the resummation of fermion bubbles before the other diagrams, which are subleading.

To some extent, the construction presented here is a simple application of the general theory of Parisi, however formulated in a new way, which singles out the conformal properties and is more suitable to the research program that we have in mind. More importantly, I generate a whole class of RG flows (marginal deformations) interpolating between the conformal fixed points and show that they satisfy a remarkable set of non-perturbative strong-weak coupling dualities, also exhibited by an exact relation between the beta function and the anomalous dimension of the composite field. The non-abelian gauge coupling is non-renormalized and has a discrete set of values. Observe that our theories do not contain a Chern-Simons term. I give a general argument proving the non-renormalization theorem, based on the trace anomaly.

I work in the euclidean framework and use a modified dimensional-regularization technique. The naive dimensional technique is indeed not applicable to the theories studied here, nor to the more familiar four-fermion models, because the dynamically generated propagator does not regularize correctly. It is necessary to add a peculiar non-local term $\mathcal{L}_{\rm non\,loc}$ to the classical lagrangian. This term does not generate new renormalization constants and is evanescent, therefore formally absent in $D = 3$.

I start from the four-fermion model, written in terms of an auxiliary field $\sigma$:

$$\mathcal{L}_N = \sum_{i=1}^{N} \bar{\psi}^i \left(\partial\!\!\!/ + \lambda\sigma\right) \psi^i + \frac{1}{2} M\sigma^2 \,. \tag{1}$$

This theory was constructed rigorously in [23], where the existence of an interacting UV fixed point was established. A detailed study can be found in [19]. There are two phases, and the chiral symmetry can be dynamically broken. The $\sigma$-field equation gives $\sigma = -\lambda\bar{\psi}\psi/M$, whence the name "composite boson" for $\sigma$.

The theory is well-defined also if we set $M = 0$. The model

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}^i \left(\partial\!\!\!/ + \lambda\sigma\right) \psi^i \tag{2}$$

is conformal both at the classical and quantum levels, as we now prove. We call it the $\sigma_N$ conformal field theory. At the classical level no scale is present. The renormalized lagrangian has the form

$$\mathcal{L} = Z_\psi \bar{\psi}\partial\!\!\!/\psi + \lambda_{\rm B} Z_\sigma^{1/2} Z_\psi \bar{\psi}\sigma\psi + \mathcal{L}_{\rm non\,loc} \,.$$

$\mathcal{L}_{\rm non\,loc}$ denotes the evanescent term to be discussed below. No $\sigma^3$-term is generated by renormalization, because of the symmetry $x_1 \rightarrow -x_1$, $\psi \rightarrow \gamma_1\psi$, $\sigma \rightarrow -\sigma$. The quadratic terms in $\sigma$ are also absent: (i) the mass term $M\sigma^2$ is not generated, because it is absent in the classical lagrangian and we can choose a subtraction scheme such that the cut-off appears only logarithmically in the quantum action; (ii) no local kinetic term for $\sigma$ can be generated, since the field $\sigma$ has dimension 1 in $D = 3$.

In general, the bare coupling can be written as $\lambda_{\rm B} = \lambda Z_\lambda \mu^{\varepsilon/2}$. However, the number of independent renormalization constants is equal to the number of independent fields and therefore we can interpret two $Z$'s as the wave-function renormalization constants of $\psi$ and $\sigma$, and set $Z_\lambda \equiv 1$. This ensures that $\beta_\lambda \equiv 0$ in $D = 3$ and proves that the theory is conformal also at the quantum level. At the level of the trace anomaly, conformality (i.e. $\Theta \equiv 0$) follows from the fact that all local dimension-3 operators are proportional to the field equations.



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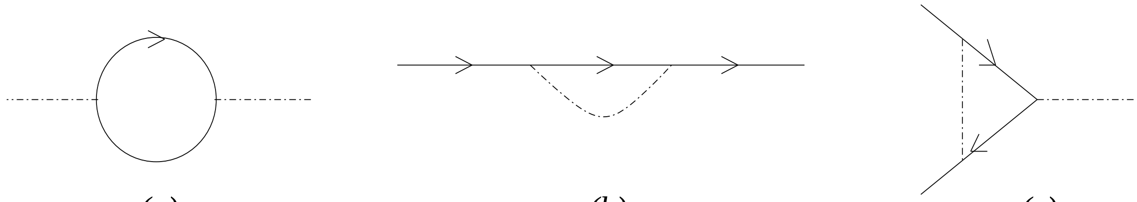

(a) (b) (c)

**Figure 1:** Leading diagram and first subleading corrections.

The dynamical $\sigma$ kinetic term is generated by diagram 1$a$, which, expanded around three dimensions, gives

$$\begin{aligned}\text{diagram } 1a &= -\frac{N\lambda_{\mathrm{B}}^2}{(4\pi)^{D/2}}\frac{\Gamma\left(2-D/2\right)\Gamma^2\left(D/2-1\right)}{\Gamma\left(D-2\right)}(k^2)^{D/2-1} \\ &= -\frac{\lambda_{\mathrm{B}}^2 N}{8}(k^2)^{(1-\varepsilon)/2} + \mathcal{O}(\varepsilon)\,. \end{aligned} \tag{3}$$

We fix the normalization with

$$\lambda^2 N = 8 + \mathcal{O}\left(\frac{1}{N}\right), \tag{4}$$

in $D=3$ and find, in momentum space,

$$\Gamma_{\mathrm{kin}}[\sigma] = \frac{1}{2}|\sigma(k)|^2\mu^{\varepsilon}(k^2)^{(1-\varepsilon)/2} + \frac{1}{2}M\sigma^2\,. \tag{5}$$

From the diagrammatic point of view, the reader might find it easier to imagine that the mass $M$ is still non-zero, but small, and set it to zero at the end. In particular, at $M \neq 0$ it is immediate to resum the geometric series of the bubbles of type ($a$) (see figure 1). After inverting the $\sigma$ kinetic term and finding the propagator

$$\langle\sigma(k)\ \sigma(-k)\rangle = \frac{1}{M}\sum_{L=0}^{\infty}(-1)^L\frac{\mu^{L\varepsilon}(k^2)^{L(1-\varepsilon)/2}}{M^L} = \frac{1}{\mu^{\varepsilon}(k^2)^{(1-\varepsilon)/2}+M}\,, \tag{6}$$

$M$ can be freely set to 0, which we assume from now on. We see that the propagator of the $\sigma$-field is proportional to $1/\sqrt{k^2}$ in $D=3$. The propagator (6), however, does not regularize the theory properly, because it goes to zero too slowly at high energies. This fact becomes apparent in the calculations of the subleading corrections. Consider the example of diagram 1$b$, where the dashed line is meant to be the $\sigma$-propagator (6). The integral

$$\int\frac{\mathrm{d}^{3-\varepsilon}p\quad(\not{p}+\not{k})}{(p+k)^2(p^2)^{(1-\varepsilon)/2}}$$

produces a $\Gamma(0)$. The same holds for diagram 1$c$. This phenomenon is very general and concerns theories of composite bosons in every dimension, and in particular the logarithmically trivial $D=4$ four-fermion models considered by Wilson in [27]. We conclude that the naive dimensional-regularization procedure fails to regularize our theories.

The problem can be cured by giving a classical, but evanescent, kinetic term to the composite field $\sigma$, which at the leading order reads

$$\mathcal{L}_{\text{non loc}} = \frac{1}{2}|\sigma(k)|^2\sqrt{k^2}\left[1-\frac{\lambda_{\mathrm{B}}^2 N}{8}(k^2)^{-\varepsilon/2}\right]. \tag{7}$$

$\mathcal{L}_{\text{non loc}}$ is renormalization-group invariant. This requirement is essential for an easier study of the theory. The new $\Gamma_{\text{kin}}$ is obtained by adding (7) to the old one, namely (5), henceforth producing the desired high-energy behaviour:

$$\Gamma'_{\text{kin}}[\varphi] = \frac{1}{2}|\sigma(k)|^2\sqrt{k^2}\,, \tag{8}$$

which regularizes the theory correctly. It is easy to go through the usual proofs of renormalizability and locality of the counterterms with the improved dimensional technique.

In $x$-space we find, in $D=3$,

$$\langle\sigma(x)\ \sigma(0)\rangle = \frac{1}{2\pi^2|x|^2}\,. \tag{9}$$

This two-point function is intrinsically non-perturbative, since it equals the two-point function of an elementary scalar field with anomalous dimension $+1/2$.

The field $\psi$ has dimension $(D-1)/2$ and (7) attributes exactly the same dimension to $\sigma$. Taking the $\mu$ -derivative of the equation $\lambda_{\mathrm{B}} = \lambda\mu^{\varepsilon/2}$, we get

$$\beta(\lambda) = \frac{\mathrm{d}\lambda}{\mathrm{d}\ln\mu} = -\frac{\varepsilon}{2}\lambda\,.$$

Integrating the defining relation

$$\gamma_\sigma(\lambda) = \frac{1}{2}\frac{\mathrm{d}\ln Z_\sigma(\lambda,\varepsilon)}{\mathrm{d}\ln\mu}\,,$$

we get the $\sigma$-wave-function renormalization constant [24, 25]:

$$Z_\sigma(\lambda,\varepsilon) = \exp\left(-\frac{4}{\varepsilon}\int_0^\lambda \frac{\gamma_\sigma(\lambda')}{\lambda'}\mathrm{d}\lambda'\right).$$

We assume that we work in the minimal subtraction scheme. We want to find a closed expression for $\mathcal{L}_{\text{non loc}}$ that properly includes the subleading corrections. The requirements are that $\mathcal{L}_{\text{non loc}}$ be renormalization-group invariant and evanescent. An expression for $\mathcal{L}_{\text{non loc}}$ satisfying these properties reads, in momentum space,

$$\mathcal{L}_{\text{non loc}} = \frac{1}{2}|\sigma(k)|^2\sqrt{k^2}\left[1-\frac{\lambda_{\mathrm{B}}^2 N}{8}(k^2)^{-\varepsilon/2}\right]\exp\left(\frac{4}{\varepsilon}\int_\lambda^{\lambda_{\mathrm{B}}(k^2)^{-\varepsilon/4}} \frac{\gamma_\sigma(\lambda')}{\lambda'}\mathrm{d}\lambda'\right).$$

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This formula is essentially unique, the alternatives differring by scheme redefinitions. Renormalization-group invariance is exhibited by rewriting $\mathcal{L}_{\text{non loc}}$ as

$$\mathcal{L}_{\text{non loc}} = \frac{1}{2}|\sigma_{\text{B}}(k)|^2\sqrt{k^2}\left[1-\frac{\lambda_{\text{B}}^2 N}{8}(k^2)^{-\varepsilon/2}\right]\exp\left(\frac{4}{\varepsilon}\int_0^{\lambda_{\text{B}}(k^2)^{-\varepsilon/4}}\frac{\gamma_\sigma(\lambda')}{\lambda'}\text{d}\lambda'\right),$$

where $\sigma_{\text{B}} = \sigma Z_\sigma^{1/2}$. It is easy to prove that $\mathcal{L}_{\text{non loc}}$ is zero in $D = 3$. Indeed, we have in the $\varepsilon \to 0$ limit:

$$\mathcal{L}_{\text{non loc}} \sim \frac{1}{2}|\sigma(k)|^2\sqrt{k^2}\left[1-\frac{\lambda_{\text{B}}^2 N}{8}(k^2)^{-\varepsilon/2}\right]\left(\frac{\mu^2}{k^2}\right)^{\gamma_\sigma(\lambda)} \to 0\,.$$

A straightforward application of the Callan-Symanzik equations shows that the $\sigma$-two-point function has the form

$$\Gamma_{\sigma\sigma} = A(\lambda)\sqrt{k^2}\left(\frac{\mu^2}{k^2}\right)^{\gamma_\sigma(\lambda)}, \tag{10}$$

or, in $x$-space,

$$\langle\sigma(x)\ \sigma(0)\rangle = \frac{A'(\lambda)}{|x|^{2+2\gamma_\sigma(\lambda)}\mu^{2\gamma_\sigma(\lambda)}}\,. \tag{11}$$

The numerical coefficients $A(\lambda)$ and $A'(\lambda)$ do not have here a direct physical meaning, because they are scheme-dependent and can be changed by redefining $\mu$.

Formulas (10) and (11) have the expected form for a conformal field theory. A non-vanishing anomalous dimension $\gamma_\sigma(\lambda)$ proves that the theory is interacting. We now calculate $\gamma_\sigma(\lambda)$ to the lowest order. We find, from diagrams $1b$ and $1c$,

$$Z_\psi = 1 - \frac{\lambda^2}{6\pi^2\varepsilon}\,, \qquad Z_\sigma = 1 + \frac{4\lambda^2}{3\pi^2\varepsilon} \tag{12}$$

and the anomalous dimensions are

$$\gamma_\psi = \frac{2}{3N\pi^2}\,, \qquad \gamma_\sigma = -\frac{16}{3N\pi^2}\,.$$

These values are in agreement with the calculations of [19] (they can be checked using the formulas (2.35a-b) of that paper, after replacing $N$ with $N/2$, since the authors of [19] use doublets of complex spinors). Higher-order corrections have been studied by Gracey in refs. [26]. It is important to remark that $\gamma_\sigma$ is negative. A negative anomalous dimension for the composite boson is not in contradiction with unitarity. We have already observed that the uncorrected $\sigma$-dimension is 1/2-larger than the minimum. The unitarity bound is therefore $d_\sigma = 1+\gamma_\sigma > 1/2$ or $\gamma_\sigma > -1/2$, so that $\gamma_\sigma$ is allowed to have negative values in three dimensions. Observe that $\gamma_\psi$ is instead positive and could not be otherwise for a similar reason. To the first subleading order we have therefore the $x$-space correlator

$$\langle\sigma(x)\ \sigma(0)\rangle = \frac{1}{2\pi^2|x|^{2-32/(3N\pi^2)}}\,.$$

Summarizing, we have formulated, via a large-$N$ expansion and an improved dimensional-regularization technique, a class of interacting conformal field theories in three dimensions. These theories are in general strongly coupled. They become weakly coupled for $N$ large, and free for $N=\infty$.

Now, we want to define renormalization-group flows interpolating between the $\sigma_{N+M}$ and the $\sigma_N$ conformal field theories. Let us consider the lagrangian

$$\mathcal{L}_{NM}=\sum_{i=1}^{M}\bar{\chi}^i\left(\partial\!\!\!/+g\sigma\right)\chi^i+\sum_{i=1}^{N}\bar{\psi}^i\left(\partial\!\!\!/+\lambda\sigma\right)\psi^i\,. \tag{13}$$

Here we expand perturbatively in $g$, or actually $\bar{g}=g/\lambda$. For $\bar{g}=0$ we have the $\sigma_N$ model plus $M$ free fermions. For $\bar{g}=1$ we have the $\sigma_{N+M}$ model. It is therefore natural to expect that the coupling $\bar{g}$ interpolates between the two fixed points. We can show that there is a non-trivial beta function by studying the first perturbative corrections. We combine the small-$\bar{g}$ expansion with the large-$N$ expansion. We also assume that $\bar{g}^2M/N\ll 1$. Since $\bar{g}$ varies from 0 to 1, this means that $M$ is much smaller than $N$. The renormalized lagrangian reads

$$\mathcal{L}_R=\sum_{i=1}^{M}Z_\chi\bar{\chi}^i\left(\partial\!\!\!/+gZ_{\bar{g}}Z_\sigma^{1/2}\sigma\right)\chi^i+\sum_{i=1}^{N}Z_\psi\bar{\psi}^i\left(\partial\!\!\!/+\lambda Z_\sigma^{1/2}\sigma\right)\psi^i+\mathcal{L}_{\text{non loc}}$$

and the evanescent, renormalization-group invariant, non-local kinetic term reads, in the general case:

$$\mathcal{L}_{\text{non loc}}=\frac{1}{2}|\sigma(k)|^2\sqrt{k^2}\left[1-\frac{\lambda_{\rm B}^2N}{8}(k^2)^{-\varepsilon/2}\right]\exp\left(-2\int_{\ln\mu}^{\ln\sqrt{k^2}}\gamma_\sigma(\ln\mu')\,\mathrm{d}\ln\mu'\right).$$

From the results (12) we easily get, to the lowest order,

$$\begin{aligned}Z_\psi&=1-\frac{4\mu^{-\varepsilon}}{3N\pi^2\varepsilon}\,, &\qquad Z_\chi&=1-\frac{4\bar{g}^2\mu^{-\varepsilon}}{3N\pi^2\varepsilon}\,,\\ Z_\sigma&=1+\frac{32\mu^{-\varepsilon}}{3N\pi^2\varepsilon}\,, &\qquad Z_{\bar{g}}&=1+\frac{16\left(\bar{g}^2-1\right)\mu^{-\varepsilon}}{3N\pi^2\varepsilon}\,.\end{aligned} \tag{14}$$

We therefore obtain

$$\beta_{\bar{g}}=\frac{16}{3N\pi^2}\bar{g}\left(\bar{g}^2-1\right)+\mathcal{O}\left(\frac{\bar{g}}{N^2},\frac{\bar{g}^5}{N}\right) \tag{15}$$

and conclude that the $\sigma_N$ model plus $M$ decoupled fermions is the UV limit of the flow and the $\sigma_{N+M}$ point is the IR limit. Remarkably, the first orders in $\bar{g}$ single out correctly both fixed points. This means that, presumably, every truncation of the perturbative expansion of $\beta_{\bar{g}}$ factorizes the expected $\bar{g}\left(\bar{g}^2-1\right)$. We show below that this is indeed the case. The theories with couplings $\bar{g}$ and $-\bar{g}$ are clearly equivalent.

The flows (13) satisfy a natural strong-weak coupling duality, associated with the replacement $\bar{g} \leftrightarrow 1/\bar{g}$, $N \leftrightarrow M$. The dual flow interpolates from the UV $\sigma_M$ model with $N$ free fermions to the IR $\sigma^{N+M}$ model. Pairs of dual flows have the IR limits in common. Finally, the self-dual flow has $N = M$. We immediately realize that the $\sigma_M$ model plus $N$ free fermions is the fixed point at $\bar{g} = \infty$. It is natural to conjecture that the points $\bar{g} = 0, 1, \infty$ are all the fixed points of the exact beta function. The dual flows are plotted in figur 2.

The mentioned duality and fixed points are non-perturbative properties of the flows and are self-evident from the construction. We have already seen that, unexpectedly, the lowest order beta function (15), calculated for $\bar{g} \ll 1$, vanishes at $\bar{g} = 1$. What is even more astonishing is that, with a little improvement, the beta function vanishes also at $\bar{g} = \infty$ and satisfies the mentioned duality exactly. To see this, let us relax the assumption $\bar{g}^2 M/N \ll 1$, so that $M$ and $N$ can be of the same order. Diagram $1a$ is proportional to $N + M\bar{g}^2$. The above formulas can be corrected replacing $N$ by $N + M\bar{g}^2$. In particular, the lowest-order beta function (15) becomes

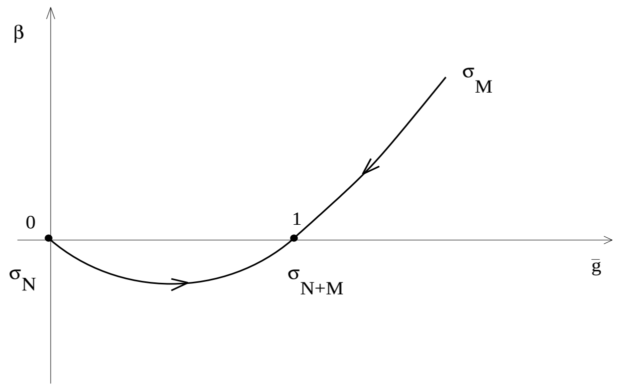


**Figure 2:** Beta function of dual RG flows.

$$\beta_{\bar{g}} = \frac{16}{3\pi^2} \frac{\bar{g}\,(\bar{g}^2 - 1)}{(N + M\bar{g}^2)},$$

and does satisfy the $\bar{g} \leftrightarrow 1/\bar{g}$, $N \leftrightarrow M$ duality, because

$$\beta_{1/\bar{g}} = \frac{16}{3\pi^2} \frac{1/\bar{g}\,(1/\bar{g}^2 - 1)}{(M + N/\bar{g}^2)}.$$

The remarkable perturbative features that we have outlined are explained by an exact relation between the beta function and the anomalous dimension of $\sigma$, that we now derive. This formula is a sort of three-dimensional analogue of certain common formulas in four-dimensional supersymmetric theories, such as the NSVZ beta function [8], or the beta function of the superpotential coupling. We stress that in three dimensions we do not need supersymmetry for this.

We write the renormalized lagrangian in a manifestly dual form:

$$\mathcal{L}_R = \sum_{i=1}^{M} V(g, M; \lambda, N; \varepsilon)\bar{\chi}^i \left(\not{\partial} + gU(g, M; \lambda, N; \varepsilon)\sigma\right) \chi^i + \\ + \sum_{i=1}^{N} V(\lambda, N; g, M; \varepsilon)\bar{\psi}^i \left(\not{\partial} + \lambda U(\lambda, N; g, M; \varepsilon)\sigma\right) \psi^i.$$

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We have

$$Z_\chi = V(g,M;\lambda,N;\varepsilon)\,, \qquad Z_\psi = V(\lambda,N;g,M;\varepsilon)\,,$$
$$Z_\sigma = U^2(\lambda,N;g,M;\varepsilon)\,, \qquad Z_g = \frac{U(g,M;\lambda,N;\varepsilon)}{U(\lambda,N;g,M;\varepsilon)}\,,$$

and find

$$\beta_{\bar g} = \bar g\,(\gamma_\sigma - \tilde\gamma_\sigma) \equiv \bar g\left[\gamma_\sigma(\bar g, M; N) - \gamma_\sigma\left(\frac{1}{\bar g}, N; M\right)\right]. \tag{16}$$

Observe that $\gamma_\sigma(1,M;N) = \gamma_\sigma(M+N)$. We can immediately check the duality of the exact beta function:

$$\beta_{1/\bar g} = \frac{1}{\bar g}\left[\gamma_\sigma\left(\frac{1}{\bar g}, N; M\right) - \gamma_\sigma(\bar g, M; N)\right]. \tag{17}$$

The beta function vanishes at the fixed points $\bar g = 0, \infty$ and the solutions of

$$\gamma_\sigma(\bar g, M; N) = \gamma_\sigma\left(\frac{1}{\bar g}, N; M\right). \tag{18}$$

Using the fact that $\gamma_\sigma(1,M;N) = \gamma_\sigma(M+N)$ we know that $\bar g = 1$ is a solution. We expect that this is the unique solution of the condition (18).

The trace anomaly reads

$$\Theta = \beta_{\bar g}\sigma \sum_{i=1}^{M} \bar\chi^i \chi^i \equiv \beta_{\bar g}\mathcal{O}$$

and, correctly, does not vanish using the field equations.

More generally, we can consider the model

$$\mathcal{L} = \sum_{i=1}^{k}\sum_{j=1}^{N_i} V(\lambda_i, N_i; \lambda, N; \varepsilon)\bar\psi^j_{(i)}\left(\partial\!\!\!/ + \lambda_i U(\lambda_i, N_i; \lambda, N; \varepsilon)\sigma\right)\psi^j_{(i)}\,,$$

where the argument $\lambda, N$ in $(\lambda_i, N_i; \lambda, N; \varepsilon)$ refers to the set of couples $\lambda_j, N_j$ with $j \neq i$. Clearly, $V(\lambda_i, N_i; \lambda, N; \varepsilon)$ and $U(\lambda_i, N_i; \lambda, N; \varepsilon)$ are symmetric with respect to the exchanges $\lambda_j, N_j \leftrightarrow \lambda_l, N_l$ with $j, l \neq i$. Choosing $\lambda_k$ to be of order unity and all the other $\lambda$'s small, we have

$$\begin{aligned}
\beta_i &= \bar\lambda_i\left(\gamma_\sigma - \tilde\gamma_\sigma^{(i)}\right)\\
&\equiv \bar\lambda_i\Bigg[\gamma_\sigma(\bar\lambda_1, N_1; \ldots \bar\lambda_i, N_i \ldots; \bar\lambda_{k-1}, N_{k-1}; N_k) -\\
&\qquad - \gamma_\sigma\left(\frac{\bar\lambda_1}{\bar\lambda_i}, N_1; \ldots \frac{1}{\bar\lambda_i}, N_k \ldots; \frac{\bar\lambda_{k-1}}{\bar\lambda_i}, N_{k-1}; N_i\right)\Bigg],\\
\gamma_\sigma &= \gamma_\sigma(\bar\lambda_1, N_1; \ldots; \bar\lambda_{k-1}, N_{k-1}; N_k) = \frac{\mathrm{d}\ln U(\lambda_k, N_k; \lambda, N; \varepsilon)}{\mathrm{d}\ln\mu},
\end{aligned}$$

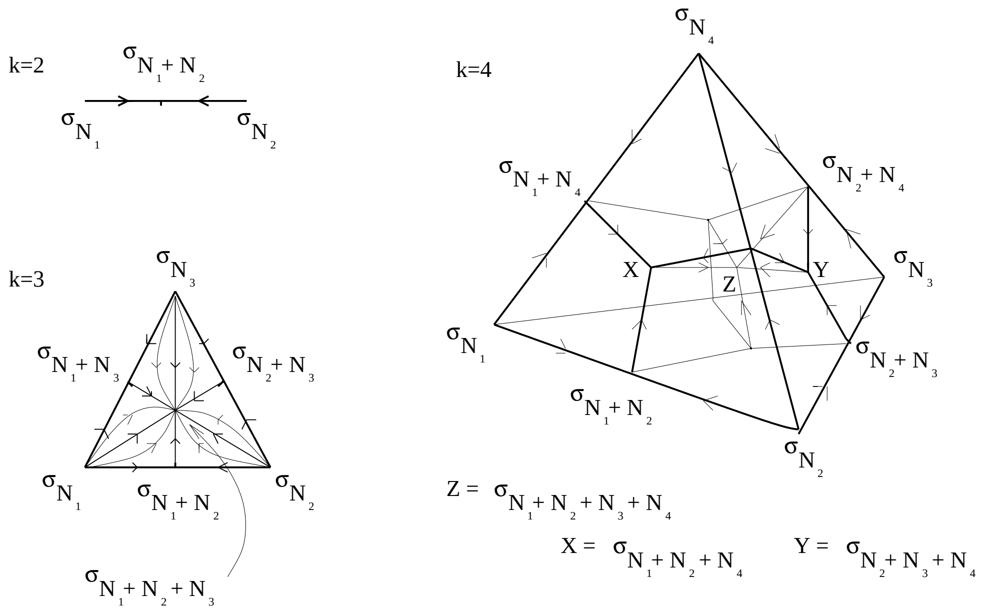


**Figure 3:** Polyhedrical RG patterns interpolating between pairs of $\sigma_N$ models.

where $\bar{\lambda}_i = \lambda_i/\lambda_k$ and $i = 1, \ldots, k-1$. The list of fixed points is obtained by assigning the values $0, 1, \infty$ to $\bar{\lambda}_1, \ldots, \bar{\lambda}_{k-1}$ in all possible ways, keeping in mind that when some $\bar{\lambda}$'s are infinite, it is immaterial whether the finite $\bar{\lambda}$'s are 0 or 1. In total, we have $2^k - 1$ fixed points, corresponding to the models $\sigma_{\sum_s N}$ for all possible subsets $s$ of $(N_1, \ldots, N_k)$. The flows are naturally associated with the regular polyhedron having $k$ faces in $k-1$ dimensions (triangle for $k=3$, tetrahedron for $k=4$, etc.) and the dualities are symmetries of this polyhedron. The RG patterns for $k = 2, 3, 4$ are illustrated in figure 3.

The trace anomaly reads

$$\Theta = \sigma \sum_{i=1}^{k-1} \beta_i \sum_{j=1}^{N_i} \bar{\psi}^j_{(i)} \psi^j_{(i)} \,.$$

Flows interpolating between the UV $\sigma_{N+M}$ and IR $\sigma_N$ fixed points can be obtained by giving mass to $M$ fermions. For the general purposes mentioned in the introduction, these flows are less interesting than the pure RG flows, which preserve conformality at the classical level and run only due to the dynamical scale $\mu$ [10]. In some cases, nevertheless, such as the vector models constructed below, giving masses to the fermions seems the only simple way to interpolate between pairs of fixed points, because a non-renormalization theorem forbids the running of the gauge coupling constant.

Vector four-fermion models were also considered in [17]. Here I make a set of observations on the non-abelian composite gauge bosons, and prove that their coupling constant is quantized and non-renormalized. The abelian coupling, instead, is non-renormalized, but can take arbitrary values.

We start from the four-fermion model defined by the lagrangian

$$\mathcal{L} = \overline{\psi}^i \partial\!\!\!/ \psi^i + \frac{\lambda^2}{2M}\left[(\overline{\psi}^i \gamma_\mu \psi^i)^2\right], \tag{19}$$

to which we associate the conformal field theory

$$\mathcal{L} = \bar{\psi}^i \left(\partial\!\!\!/ + i\lambda A\!\!\!/\right) \psi^i \,.$$

The vector $A_\mu$ becomes dynamical at the quantum level and the resulting conformal theory is interacting. The diagram $1a$ generates the $A_\mu$-propagator at the leading order in the large-$N$ expansion and its kinetic term in the quantum action reads, in momentum space and coordinate space, respectively:

$$\Gamma_{\text{kin}}[A] = \frac{1}{2}\frac{\lambda^2 N}{16} A_\mu(k) A_\nu(-k) \frac{k^2\delta_{\mu\nu} - k_\mu k_\nu}{\sqrt{k^2}} = \frac{1}{4}\frac{\lambda^2 N}{16} F_{\mu\nu} \frac{1}{\sqrt{-\Box}} F_{\mu\nu}\,, \tag{20}$$

$F_{\mu\nu}$ denoting the field strength. At the leading order we set again

$$\frac{\lambda^2 N}{16} = 1, \qquad \lambda_{\rm B} = \lambda \mu^{\varepsilon/2}\,.$$

Since the U(1) currents are conserved, the subleading corrections can only change the coefficient of the quadratic term in (20), but cannot change the dimension of the vector. There is, nevertheless, a non-vanishing anomalous dimension for the fermion fields, calculable from diagram $1b$ or, alternatively, from $1c$. We find, using an analogue of the Feynman gauge,

$$\gamma_\psi = \frac{4}{3N\pi^2} + \mathcal{O}\left(\frac{1}{N^2}\right), \qquad \gamma_A \equiv 0\,.$$

Finally, the non-local lagrangian kinetic term of the vector reads

$$\mathcal{L}_{\text{non loc}} = \frac{1}{2} A_\mu(k) A_\nu(-k) \frac{k^2\delta_{\mu\nu} - k_\mu k_\nu}{\sqrt{k^2}} \left[1 - \frac{\lambda_{\rm B}^2 N}{16}(k^2)^{-\varepsilon/2}\right]$$

and does not need subleading corrections, since $\gamma_A = 0$.

It is straightforward to construct conformal field theories with non-abelian gauge fields, using the same method. The lagrangian

$$\mathcal{L} = \bar{\psi}^i \left[\delta_{ij} \partial\!\!\!/ + i\lambda A\!\!\!/^a T^a_{ij}\right] \psi^j$$

generates a gauge-field quantum action

$$\Gamma_{\text{kin}}[A] = \frac{1}{4}\frac{\lambda^2 C(T) N_f}{16} F^a_{\mu\nu} \frac{1}{\sqrt{-\Box}} F^a_{\mu\nu}\,,$$

for the field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \lambda f^{abc} A^b_\mu A^c_\nu\,.$$

With a natural normalization convention for the action we see that the gauge coupling is discretized and equals

$$g^2 = \frac{16}{C(T)N_f} + \mathcal{O}\left(\frac{1}{N_f^2}\right).$$

We have the freedom to change the representation $T$ and $N_f$, but since the set of unitary representations is denumerable, the non-abelian gauge coupling can take only discrete values. It remains to be seen whether we can interpolate between two models with different values of the gauge coupling by means of an RG flow. However, a non-renormalization theorem forbids this. Consider the trace anomaly $\Theta$. The term responsible for the running of the gauge coupling should be proportional to the gauge beta function multiplied by a non-trivial dimension-3, local, gauge-invariant operator. However, there exists no such operator in the gauge sector, all candidates being proportional to the field equations. The usual term $F_{\mu\nu}^2$ has dimension 4, while terms proportional to $\not{D}\psi$ are trivial.

Other interesting conformal field theories are given by the gauged $\sigma_N$ models

$$\mathcal{L} = \bar{\psi}^i \left[\delta_{ij}\not{\partial} + i\lambda \not{A}^a T^a_{ij} + \lambda' \sigma \delta_{ij}\right] \psi^j .$$

RG flows such as

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}^i \left[\delta_{ij}\not{\partial} + i\lambda \not{A}^a T^a_{ij} + \lambda' \sigma \delta_{ij}\right] \psi^j + \sum_{i=1}^{M} \bar{\chi}^i \left[\delta_{ij}\not{\partial} + i\lambda \not{A}^a R^a_{ij} + g \sigma \delta_{ij}\right] \chi^j$$

do not change the gauge-coupling, by the non-renormalization theorem proved above, but only the $\sigma$ coupling. The patterns of their RG flows are similar to the RG patterns of the non-gauged $\sigma_N$ models, with the only difference that the duality symmetries involve also exchanges of the representations, such as $R \leftrightarrow T$, etc.

A non-abelian coupling can take arbitrary values, but the non-renormalization theorem applies. Consider for example

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}^i \left[\not{\partial} + i\lambda \not{A}\right] \psi^i + \sum_{i=1}^{M} \bar{\chi}^i \left[\not{\partial} + i\lambda' \not{A}\right] \chi^i .$$

Here the trace anomaly is still identically zero and the theory is conformal for arbitrary $\lambda$ and $\lambda'$. We have

$$\Gamma_{\mathrm{kin}}[A] = \frac{1}{4}\frac{\lambda^2 N + \lambda'^2 M}{16} F_{\mu\nu} \frac{1}{\sqrt{-\Box}} F_{\mu\nu} .$$

We can reduce to the original vector four-fermion model (19) by means of a relevant deformation. A mass perturbation, such as $MA_\mu^2/2$, produces the propagator

$$\langle A^a_\mu(k)\ A^b_\nu(-k)\rangle = \frac{\delta^{ab}\sqrt{k^2}}{k^2 + M\sqrt{k^2}} \left(\delta_{\mu\nu} + \frac{k_\mu k_\nu}{M\sqrt{k^2}}\right).$$



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The behaviour $k_\mu k_\nu/(Mk^2)$ at large momenta is not dangerous if the current is conserved [17], which happens for abelian fields. The situation is similar to quantum electrodynamics in four dimensions, where the photon can be given a mass without spoiling the renormalizability. With non-abelian gauge fields we have to advocate a symmetry-breaking mechanism. We consider

$$\mathcal{L} = \bar{\psi}^i \left(\delta_{ij}\partial\!\!\!/ + i\lambda A\!\!\!/^a T^a_{ij}\right)\psi^j + |D_\mu\varphi|^2 + V(|\varphi|) + \Lambda\bar{\psi}\psi\,\bar{\varphi}\varphi + \Lambda'\bar{\psi}T^a\psi\,\bar{\varphi}R^a\varphi\cdots$$

and assume that the potential $V(|\varphi|)$ is such that the scalar field has an expectation value $\langle|\varphi|\rangle = M^{1/2}$. We know that the theory is renormalizable in the large-$N$ expansion. We can integrate the vector field out by solving its field equation. For simplicity we write the formulas in the abelian case, although the mechanism is not strictly necessary there. We have

$$A^a_\mu = -\frac{i}{2\lambda\,|\varphi|^2}\left(\overline{\psi}\gamma_\mu\psi - \bar{\varphi}\partial_\mu\varphi + \partial_\mu\varphi\bar{\varphi}\right) = -\frac{i\overline{\psi}\gamma_\mu\psi}{\lambda|M^{1/2}+\eta|^2} - \frac{1}{\lambda}\partial_\mu\theta\,,$$

having set $\varphi = \mathrm{e}^{i\theta}(M^{1/2}+\eta)/\sqrt{2}$. The Goldstone boson $\theta$ is gauged away as usual and we remain with

$$\mathcal{L} = \bar{\psi}^i\partial\!\!\!/\psi^i - \frac{\left(\overline{\psi}^i\gamma_\mu\psi^i\right)^2}{2M\left|1+\eta/\sqrt{M}\right|^2} + \frac{1}{2}\left(\partial_\mu\eta\right)^2 + V(\eta) + \frac{\Lambda}{2}\bar{\psi}\psi\left|1+\frac{\eta}{\sqrt{M}}\right|^2. \tag{21}$$

When the mass of $\eta$ is very large, (19) is recovered exactly. By construction, the theory (21) is renormalizable, although this is not evident in the final form. Since the limit of large $\eta$-mass can be taken at $M$ fixed, we see that (19) is also renormalizable.

A more direct way to get to (19) is by replacing $V(|\varphi|)$ with $i\alpha(\bar{\varphi}\varphi - M)$, such as in the $S_{N-1}$ non-lineal $\sigma$-model [20]. The field $\alpha$ is dynamical and acquires a propagator proportional to $\sqrt{k^2}$, which is however compatible with power counting. In this case, however, we need to take a large-$N$ limit also in the number of $\varphi$ components.

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